JOURNAL OF APPROXIMATION THEORY 9, 367–372 (1973)

Convergence Rates for Exponential Interpolation Series

JOHN W. LAYMAN

Department of Mathematics, Virginia Polytechnic and State University, Blacksburg, Virginia 24061

Received November 15, 1971

Communicated by John R. Rice

An analysis of the rate of convergence is made for the interpolation series based on the biorthogonal system $\binom{\Delta}{n}f(0)$ and $e_n(z) = \Delta^n x^z|_{x=1}$, which was recently shown to be convergent for certain entire functions of exponential type. An error bound is obtained which is shown to vary as a negative power of the number of terms in the partial sum. Comparison is made with numerical calculations in a few simple cases and certain practical applications are mentioned.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n/n!$ be entire and of exponential type, and let $F(w) = \sum_{n=0}^{\infty} a_n/w^{n+1}$. Let $D(f)$ be the set consisting of the singular points of $F(w)$ and the points exterior to the domain of F. The following result on the expansion of analytic functions has recently been established [3] using the method of kernel expansion in the Pólya representation as expounded, for example, in Buck [2] or Boas and Buck [1].

THEOREM 1. *If $D(f)$ lies in the strip $|\operatorname{Im}(w)| < \pi/2$, then $f(z)$ admits the convergent exponential interpolation series expansion*

$$f(z) = \sum_{n=0}^{\infty} e_n(z) \binom{\Delta}{n} f(0), \tag{1}$$

for all z, where $e_n(z)$ is the exponential polynomial

$$e_n(z) = \Delta^n x^z|_{x=1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+1-k)^z.$$

(Several of the $e_n(z)$ are: $e_0(z) = 1$, $e_1(z) = 2^z - 1$, $e_2(z) = 3^z - 2\cdot 2^z + 1,\ldots$.)

In the present paper we shall give the results of an error analysis of the approximation obtained by using the first n terms of the expansion (1). The error bound which we obtain varies as a negative power of n and is related to the set $D(f)$ defined above.

It should perhaps be pointed out here that the linear functionals $\binom{\Delta}{n} f(0)$ and the exponential polynomials $e_n(z)$ form a biorthogonal system and, hence, for any function f and any $m = 0, 1, 2, 3, \ldots$, we have

$$f(m) = \sum_{n=0}^{m+1} e_n(m) \binom{\Delta}{n} f(0).$$

Our error bound is thus not required for z equal to a positive integer or zero.

The Pólya representation mentioned above may be stated as follows: If $f(z) = \sum_{n=0}^{\infty} a_n z^n/n!$ is entire and of exponential type, then

$$f(z) = (2\pi i)^{-1} \int_\Gamma e^{zw} F(w)\, dw, \tag{2}$$

where Γ encircles $D(f)$. If the kernel expansion $e^{zw} = \sum_0^\infty U_n(z)\, g_n(w)$ holds uniformly for all w on a simple contour Γ which encircles $D(f)$ and for all z, we may integrate termwise in (2) and obtain $f(z) = \sum_0^\infty T_n(f)\, U_n(z)$, for all z, where

$$T_n(f) = (2\pi i)^{-1} \int_\Gamma g_n(w)\, F(w)\, dw.$$

In order to obtain the expansion of Theorem 1, we first expand $f(z) = z^\gamma$, γ any complex number, $-\pi/2 < \arg z < \pi/2$, into a Newton series and obtain

$$z^\gamma = \sum_0^\infty e_n(\gamma) \binom{z-1}{n}, \tag{3}$$

uniformly in any bounded region of the half plane $\operatorname{Re}(z) \geqslant \epsilon > 0$, for arbitrary γ. Setting $z \to e^w$, $\gamma \to z$ in (3), we obtain the kernel expansion

$$e^{zw} = \sum_{n=0}^\infty e_n(z) \binom{e^w - 1}{n}, \tag{4}$$

uniformly in any bounded region in the strip $|\operatorname{Im}(w)| \leqslant \delta < \pi/2$. Termwise integration now yields Theorem 1. The details may be found in [3].

Since our primary concern here is with the rate of convergence, we write (3) in the form

$$z^\gamma = \sum_{k=0}^n e_k(\gamma) \binom{z-1}{k} + r_n(z). \tag{5}$$

Nörland [4] gives the following estimate for the remainder in Newton series expansions. *If $f(z)$ is holomorphic in the half plane* $\operatorname{Re}(z) \geqslant \alpha$ *and satisfies in that half plane the inequality* $|f(\alpha + re^{i\theta})| < Ce^{r \log 2}\, (1 + r)^{\beta + \epsilon(r)}$,

$-\pi/2 \leqslant \theta \leqslant \pi/2$, *where* $\epsilon(r) \to 0$ *as* $r \to \infty$, *then the remainder* $r_n(z)$ *after* n *terms of the Newton series satisfies the inequality*

$$| r_n | \leqslant c_1 n^{\beta+\epsilon+\frac{1}{2}-\sigma} + c_2 n^{\alpha-\sigma}(\log n)^{\beta+\epsilon+\frac{1}{2}-\alpha} + c_3 n^{\alpha-\sigma}(1 + \log n)^{\beta+2-\alpha-\epsilon},$$

where $\sigma = \mathrm{Re}(z)$ *and* n *is sufficiently large.* For $f(z) = z^\gamma$ we have α arbitrarily small and positive and a straightforward calculation shows that we may assign an arbitrary negative value to β. If we take $\beta = -3$, then it is possible to show that $| r_n | \leqslant Cn^{\alpha-\sigma}$. Here and below C will represent an unknown constant which is not necessarily the same at each occurrence. We now introduce the transformation $z = e^w$ in (5) to obtain

$$e^{w\gamma} = \sum_{k=0}^{n} e_k(\gamma) \binom{e^w - 1}{n} + r_n, \tag{6}$$

where now $| r_n | < Cn^{\alpha-\sigma}$, $\sigma = \mathrm{Re}\, e^w$. If we let $w = s + it$ we have

$$| r_n | < Cn^{\alpha-e^s\cos t}. \tag{7}$$

We now change $\gamma \to z$ in (6) and substitute into (2) to obtain

$$f(z) = \sum_{k=0}^{n} e_k(z) \binom{\Delta}{k} f(0) + R_n(z),$$

where $R_n(z) = (2\pi i)^{-1} \int_\Gamma r_n(z)\, F(w)\, dw$. We have thus proved the following theorem, in which Γ encircles $D(f)$ and $w = s + it$.

THEOREM 2. *If* $f(z)$ *is entire and of exponential type and if* $D(f)$ *lies in the strip* $| \mathrm{Im}(w) < \pi/2 |$, *then* $f(z) = \sum_{k=0}^{n} e_k(z)\binom{\Delta}{n} f(0) + R_n(z)$, *where* $| R_n(z)| < C \,| \int_\Gamma n^{\alpha-e^s\cos t} F(w)\, dw |$.

By considering bounds on the quantities in the integral of Theorem 2, we obtain the following useful result.

COROLLARY. *In Theorem* 2, *the remainder satisfies the inequality*

$$| R_n | < Cn^{\alpha-e^S\cos T},$$

where $S = \inf\{s: s + it \in D(f)\}$, $T = \sup\{t: s + it \in D(f)\}$.

We now illustrate the above result by considering $f(z) = z$, in which case $F(w) = 1/w^2$ and $D(f) = \{0\}$. We have for the coefficients of the $e_n(z)$ in (1),

$$\begin{aligned}\binom{\Delta}{n} f(0) &= (1/n!)\, \Delta(\Delta - 1) \cdots (\Delta - n + 1)\, z \,|_{z=0} \\ &= \begin{cases} 0 & \text{for } n = 0, \\ (-1)^{n+1}/n & \text{for } n \geqslant 1. \end{cases}\end{aligned}$$

This gives the exponential interpolation series

$$f(z) = z = e_1(z) - \tfrac{1}{2}e_2(z) + e_3(z)/3 - \cdots, \tag{8}$$

which, according to Theorem 1, holds for all z. By the corollary to Theorem 2 with $S = T = 0$, we see that the error, after a sufficiently large number of terms of (8), satisfies the bound

$$| R_n | < Cn^{\alpha-1}, \tag{9}$$

where α is arbitrarily small and positive.

Pitts [5] has computed R_n, for the expansion in (8), to $16D$ for several values of z, obtaining meaningful results to about $n = 40$. A comparison between the calculated $| R_n |$ and the theoretical bound in (9) is provided by looking at $n | R_n |$ where, for simplicity, we have set $\alpha = 0$. For $z = \frac{1}{2}$ and $n = 10, 20, 30, 40$ we obtain $n | R_n | = 0.049, 0.036, 0.030, 0.027$, respectively.

In order to illustrate the accelerated rate of convergence, indicated by the corollary, when one has a positive value for $S = \inf\{s\colon s + it \in D(f)\}$, we may consider the following example:

$$f(z) = e^z = \sum_{n=0}^{n} \binom{e-1}{n} e_n(z). \tag{10}$$

We now have $D(f) = \{1\}$ and by the corollary must have $| R_n | < Cn^{\alpha-e}$. Comparison with the computed results for $z = \frac{1}{2}$, gives, for $n = 1, 2, 3, 4$, and 5, the values $n^e | R_n | = 0.65, 0.37, 0.061, 0.031$, and 0.019, respectively.

A direct comparison of actual errors shows that for $z = \frac{1}{2}$ the error in (10) for e^z is 0.0003 after five terms, whereas, in (8) for $f(z) = z$ the error is 0.0007 after 40 terms.

It is interesting to note that $e_n(-1)$ may be evaluated in closed form. We have $e_n(-1) = \Delta^n(1/k)|_{k=1} = (-1)^n/(n+1)$. Using $f(z) = z$ again, we have

$$-1 = \sum_{k=1}^{\infty} ((-1)^{n+1}/n)\, e_n(-1) = -\sum_{k=1}^{n} 1/k(k+1) + R_n,$$

with $| R_n | = \sum_{k=n+1}^{\infty} 1/k(k+1) = 1/(n+1)$, in complete agreement with the analytical bound (9) given by the corollary.

Apart from the theoretical interest in the existence of a nonpolynomial interpolation scheme with the linear functionals being simple difference operators, there is also a rather important practical advantage in certain applications. Consider a simple problem where it is desired to fit an analytic expression to discrete data points and assume that the data are equally spaced and that the function being fitted is known to grow at least as fast as, say, e^x. Then it is well known that such a function cannot in general be expressed as a convergent Newton series without resorting to the device of shifting the conjugate indicator diagram to the left by multiplying the function by e^{-cz} (see [1, p. 35]). The exponential series of the present paper can be used, however, and in fact by a simple difference table method which we illustrate below. Even if one wishes to utilize only a few data points and obtain an approximating Newton polynomial, the result is quite inferior to using the initial terms of the exponential series of Theorem 1.

We illustrate by tabulating $f(x) = e^x$ at $x = 0, 1, 2,$ and 3 and calculating the coefficients for both Newton and exponential interpolation. An examination of the functionals $\binom{\Delta}{n} f(0)$ shows that they can be evaluated by an appropriate extension of the usual difference table, as indicated below.

x	$f(x)$						
0	1.000(N, E)						
1	2.718	1.718(N)		1.718(E)			
2	7.389	4.671	2.953(N)	2.953	1.235	1.235(E)	
3	20.086	12.697	8.026	5.073(N)	2.210	0.885	−0.350(E)

The Newton coefficients (N) are 1.000, 1.718, 2.953, and 5.073 and the exponential series coefficients (E) are 1.000, 1.718, 1.235, −0.350, yielding the following interpolating functions:

$$\text{(Newton) } e^x \cong 1.000 + 1.718x + 1.477x(x-1) + 0.845x(x-1)(x-2),$$

$$\text{(Exponential) } e^x \cong 1.000 + 1.718(2^x - 1) + 0.618(3^x - 2^{x+1} + 1) - 0.058(4^x - 3^{x+1} + 3 \cdot 2^x - 1).$$

Each of these interpolating expressions agrees with the given data at $x = 0, 1,$ 2, and 3 to within 0.001. At $x = 0.500$, 1.500, 2.500, points midway between the given data, the situation is quite different, however, with the exponential interpolating function giving errors of 0.001, 0.003, and 0.022, respectively, whereas, the Newton polynomial gives errors of 0.155, −0.114, and 0.222, respectively. It thus appears that both theory and numerical comparison indicate the utility of the exponential interpolation series.

Acknowledgment

The author wishes to thank Mr. Wesley Darbro for conversations leading to the present form of the corollary to Theorem 2.

References

1. R. P. Boas, Jr. and R. C. Buck, "Polynomial Expansions of Analytic Functions," Springer-Verlag, Berlin, 1964.
2. R. C. Buck, Interpolation series, *Trans. Amer. Math. Soc.* **64** (1948), 283–298.
3. J. W. Layman, Expansion of analytic functions in exponential polynomials, *Proc. Amer. Math. Soc.* **22** (1969), 519–522.
4. N. E. Nörlund, "Leçons sur les séries d'interpolation," Gautier-Villars, Paris, 1926.
5. G. Pitts, M. S. Thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 1968.